Eur. Phys. J. A **31**, 750–753 (2007)
DOI 10.1140/epja/i2006-10295-1

The European Physical Journal A

Special Article – QNP 2006

Constraints on the IR behaviour of gluon and ghost propagator from Ward-Slavnov-Taylor identities

Ph. Boucaud[1], J.P. Leroy[1], A. Le Yaouanc[1], A.Y. Lokhov[2], J. Micheli[1], O. Pène[1], J. Rodríguez-Quintero[3,a], and C. Roiesnel[2]

[1] Laboratoire de Physique Théorique et Hautes Energies, Université de Paris XI, Bâtiment 211, F-91405 Orsay Cedex, France
[2] Centre de Physique Théorique de l'Ecole Polytechnique, F-91128 Palaiseau cedex, France
[3] Departamento Física Aplicada, Facultad de Ciencias Experimentales, Universidad de Huelva, E-21071 Huelva, Spain

Received: 22 December 2006
Published online: 20 March 2007 –

Abstract. We consider the constraints of the Slavnov-Taylor identity of the IR behaviour of gluon and ghost propagators and their compatibility with solutions of the ghost Dyson-Schwinger equation and with the lattice picture.

PACS. 12.38.Aw General properties of QCD (dynamics, confinement, etc.) – 12.38.Gc Lattice QCD calculations – 11.15.-q Gauge field theories – 11.15.Ha Lattice gauge theory

1 Introduction

In ref. [1] we have considered the constraints on the propagator dressing functions which can be derived from the Ward-Slavnov-Taylor identity (WSTI) —supplemented with some minimal assumptions on the analytic behaviour of the former and of the vertex form factors— and we were confronted with a contradiction between them and the ones that stem from the Dyson-Schwinger equation. The analysis of the ghost propagator Dyson-Schwinger equation seems to indicate that only a non-divergent gluon can match the lattice picture for the infrared behaviour of Landau gauge Green functions. On the other hand, WSTI seems to require that the gluon propagator diverges while the ghost dressing function should be finite and non-vanishing. In that ref. [1] we proposed, as a possible way out, that the ghost-gluon vertex function was singular (which does not contradict Taylor's theorem contrary to frequent claims). That hypothesis did not look very natural and the further work of [2] made it even less plausible.

In view of the very general validity of the WSTI, this situation is rather embarrassing and we wish to reconsider the problem. In the following, we will re-analyse the problem and clarify the working hypotheses to conclude either that the gluon propagator diverges[1] or that some of these hypoteses should fail.

[a] e-mail: jose.rodriguez@dfaie.uhu.es

[1] Although softly enough as not to contradict the apparent finiteness previously stated from lattice data.

2 Notations and main hypotheses

We use the following notations [1]:

$$\left(F^{(2)}\right)^{ab}(k^2) = -\delta^{ab}\,\frac{F(k^2)}{k^2}\,,$$
$$\left(G^{(2)}_{\mu\nu}\right)^{ab}(k^2) = \delta^{ab}\,\frac{G(k^2)}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right), \tag{1}$$

where $G^{(2)}$ and $F^{(2)}$ are, respectively, the gluon and ghost propagators, G and F are, respectively, the gluon and ghost dressing functions. The ghost-gluon vertex $\widetilde{\Gamma}_\mu(p,k;q)$ (k and $-p$ are the momenta of the incoming and outgoing ghosts and q the gluon momentum) is defined as follows:

$$\Gamma^{abc}_\mu(p,k;q) = g_0(-ip_\nu)f^{abc}\widetilde{\Gamma}_{\nu\mu}(p,k;q) = ig_0 f^{abc}\,\widetilde{\Gamma}_\mu(p,k;q)\,. \tag{2}$$

It will be also useful to define the following scalars H_1 and H_2:

$$\widetilde{\Gamma}_\mu(-q,k;q-k) = q_\mu H_1(q,k) + (q-k)_\mu H_2(q,k) \tag{3}$$

that, after applying the standard tensor decomposition [3],

$$\widetilde{\Gamma}_{\nu\mu}(p,k;q) = \delta_{\nu\mu}a(p,k;q) - q_\nu k_\mu b(p,k;q) + p_\nu q_\mu c(p,k;q) + q_\nu p_\mu d(p,k;q) + p_\nu p_\mu e(p,k;q)\,, \tag{4}$$

could be written as follows:

$$H_1(q,k) = a(-q,k;q-k) - q^2\left(b(-q,k;q-k) + d(-q,k;q-k) + e(-q,k;q-k)\right)$$
$$H_2(q,k) = q^2\left(b(-q,k;q-k) - c(-q,k;q-k)\right). \tag{5}$$

We can at this point make our first hypothesis: the scalar factors present in that decomposition are regular when one of their arguments goes to zero while the others are kept finite. Thus, we suppose that

$$a(-r, r-p; p) = a_1(p^2) + \mathcal{O}(p \cdot r)\,, \tag{6}$$

and the same for the other scalars in the particular kinematic configurations we shall encounter. We adopt the notations $a_i(p^2)$, $b_i(p^2)$, $c_i(p^2)$ and so on, where the subindex i means that their i-th argument is a zero momentum.

The most general tensorial decomposition of the three-gluon vertex, $\Gamma_{\lambda\mu\nu}$ (of course, the antisymmetric color tensor f^{abc} is factorised) is given in ref. [4]. We will be interested in the limit of one vanishing gluon momentum while the two others remain finite. Such a limit deserves a careful analysis in the framework of WST identities because of the interplay of gluon and ghost propagator singularities and those of scalar functions in the decomposition [5]. When one of the momenta is zero the three-gluon vertex reduces to (cf. ref. [3]):

$$\begin{aligned}&\Gamma_{\lambda\mu\nu}(q,-q,0) = \\ &(2\delta_{\lambda\mu}q_\nu - \delta_{\lambda\nu}q_\mu - \delta_{\nu\mu}q_\lambda)\,T_1(q^2) \\ &-\left(\delta_{\lambda\mu} - \frac{q_\lambda q_\mu}{q^2}\right) q_\nu T_2(q^2) + q_\lambda q_\mu q_\nu T_3(q^2).\end{aligned} \tag{7}$$

For our purposes here, we will only assume that the limit of one vanishing gluon momentum can be safely taken, *i.e.*[2]:

$$\Gamma_{\lambda\mu\nu}(q-r,-q,r) = \Gamma_{\lambda\mu\nu}(q,-q,0) + o(1). \tag{8}$$

3 WSTI and IR propagators

The Ward-Slavnov-Taylor [6] identity for the three-gluon function reads

$$\begin{aligned}p^\lambda \Gamma_{\lambda\mu\nu}(p,q,r) = &\frac{F(p^2)}{G(r^2)}(\delta_{\lambda\nu}r^2 - r_\lambda r_\nu)\widetilde{\Gamma}_{\lambda\mu}(r,p;q) \\ &- \frac{F(p^2)}{G(q^2)}(\delta_{\lambda\mu}q^2 - q_\lambda q_\mu)\widetilde{\Gamma}_{\lambda\nu}(q,p;r).\end{aligned} \tag{9}$$

We shall now study the behaviour when $r \to 0$ while keeping q and p finite and apply decompositions (2), (7) and the hypotheses (6), (8) to replace the vertices in eq. (9). Then, if one only retains the leading terms, STI reads

$$\begin{aligned}&T_1(q^2)\left(q_\mu q_\nu - q^2\delta_{\mu\nu}\right) + q^2 q_\mu q_\nu\; T_3(q^2) + o(1) = \\ &\frac{F(q^2)}{G(r^2)}\left[a_1(q^2)\left(r^2\delta_{\mu\nu} - r_\mu r_\nu\right)\right. \\ &\left.+\, b_1(q^2) q_\mu\left(r^2 q_\nu - (q\cdot r)r_\nu\right) + o(r^2)\right] \\ &+ \frac{F(q^2)}{G(q^2)}\left[a_3(q^2)\left(q_\mu q_\nu - q^2\delta_{\mu\nu}\right) + o(1)\right].\end{aligned} \tag{10}$$

[2] It is shown in [4], on a perturbative basis, that the vertex remains finite when one takes the limit $r \to 0$ while keeping the two other momenta fixed. Our hypothesis amounts to assuming that this result survives beyond perturbation theory.

Thus, if one multiplies both l.h.s. and r.h.s. of this eq. (10) by r_ν, we obtain:

$$\begin{aligned}&T_1(q^2)\left(q_\mu\;(q\cdot r) - q^2 r_\mu\right) \\ &+ q^2 q_\mu (q\cdot r)\; T_3(q^2) + o(r\cdot q) = \\ &\frac{F(q^2)}{G(q^2)}\; a_3(q^2)\;\left(q_\mu (q\cdot r) - q^2 r_\nu\right) + o(r\cdot q)\,,\end{aligned} \tag{11}$$

where the first term of the r.h.s. of eq. (10) vanishes because it is transverse to r_ν. Thus, by identifying both r.h.s. and l.h.s. of eq. (11), one is led to the familiar relations [3]:

$$\begin{aligned}&T_1(q^2) = \frac{F(q^2)}{G(q^2)} a_3(q^2), \\ &T_3(q^2) = 0\,.\end{aligned} \tag{12}$$

Now, let us multiply both r.h.s. and l.h.s. of eq. (10) by q_μ and apply that T_3 has been seen to be exactly 0 in eq. (12) and we obtain then

$$\begin{aligned}&\frac{F(q^2)}{G(r^2)}\; r^2 \left[\left(a_1(q^2) + q^2 b_1(q^2)\right)\right. \\ &\left.\times \left(q_\nu - \frac{(q\cdot r)}{r^2}\; r_\nu\right) + o(1)\right] = o(1)\,.\end{aligned} \tag{13}$$

Thus, if $a_1(q^2) \neq 0$ or $b_1(q^2) \neq 0$ (and, indeed, one knows from perturbation theory that at large momenta $a_1 = 1$, cf. [3,6]) eq. (9) implies

$$\lim_{r\to 0} \frac{G(r^2)}{r^2} \to \infty\,, \tag{14}$$

or, in other words, that *the gluon propagator diverges in the infrared limit.* If we stick to the commonly accepted idea that G behaves as a power in the infrared ($G(p^2) \sim (p^2)^{\alpha_G}$), then $\alpha_G < 1$ is to be concluded. Another attractive possibility would be to suppose an infrared behaviour less divergent than any power as, for instance, that of the form $G(p^2) \sim p^2 \log^\nu(p^2)$ for some positive ν. This will be considered in more detail in a forthcoming paper [5].

We can also, instead of letting $r \to 0$, study now the behaviour when $p \to 0$ of eq. (9) as is done in [3]. The dominant part of the l.h.s. of (9) reads

$$\begin{aligned}&(2\delta_{\mu\nu} p\cdot q - p_\mu q_\nu - p_\nu q_\mu)\; a_3(q^2)\frac{F(q^2)}{G(q^2)} \\ &-\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)(p\cdot q) T_2(q^2)\,,\end{aligned} \tag{15}$$

where the results in eq. (12) have been implemented. Let us now multiply both sides with q^μ and keep only the leading tems in p and one obtains

$$\begin{aligned}&(q_\nu(p\cdot q) - q^2 p_\nu) a_3(q^2) F(q^2) = \\ &(q_\nu(p\cdot q) - q^2 p_\nu) F(p^2)(a_2(q^2) - q^2 d_2(q^2)) \\ &+\mathcal{O}(p^2)\end{aligned} \tag{16}$$

that of course can be true only if $F(p^2)$ goes to some finite limit when $p^2 \to 0$ and whence, in terms of scalars,

$$F(q^2) \underset{q^2\to 0}{\sim} F(0)\, \frac{a_2(q^2) - q^2 d_2(q^2)}{a_3(q^2)}\,, \qquad (17)$$

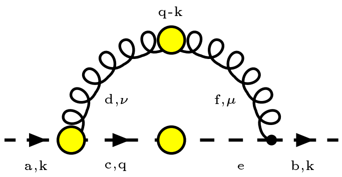
where $a_2/a_3 \to 1$ as $q^2 \to 0$ [3].

Let us repeat here that all these considerations are valid only when our regularity hypotheses about the ghost-gluon scalar factors and about the three-gluon vertex (see (6), (8)) are satisfied. *Under those hypotheses* one obtains important constraints on the gluon and ghost propagators —namely that they are divergent in the zero-momentum limit. Let us now briefly analyze the ghost propagator Dyson-Schwinger equation (GPDSE).

4 Ghost DSE: the case $\alpha_F = 0$

In a previous paper (ref. [1]) we studied all the classes of solutions for the GPDSE, that can be pictured, in diagrammatic form, as:

$$\left(F^{ab}(q^2)\right)^{-1} = \left(F^{ab}_{\text{tree}}(q^2)\right)^{-1} -$$



Let us first recall that the *unsubtracted* GPDSE is actually meaningless since the integral in its right-hand side is UV-divergent, behaving as $\int dq^2 \frac{1}{q^2}(1 + 11\alpha_s/(2\pi)\log(q/\mu))^{-35/44}$. A way out of this difficulty would be to renormalise the equation to deal properly with its UV divergencies. Instead of that, we preferred to study the following subtracted version of the bare GPDSE equation for two scales, λk and $\kappa\lambda k$ (see eq. (14) of ref. [1]) with k the external ghost momentum and κ some fixed number (< 1). λ is an extra parameter that we shall ultimately let go to 0 in order to study the infared behaviour of the GPDSE. This subtracted version of the GPDSE reads (see eq. (14) of ref. [1]):

$$\frac{1}{F(\lambda k)} - \frac{1}{F(\kappa\lambda k)} = g_B^2 N_c \int \frac{d^4 q}{(2\pi)^4} \left(\frac{F(q^2)}{q^2} \left(\frac{(k\cdot q)^2}{k^2} - q^2 \right) \times \left[\frac{G((q-\lambda k)^2) H1(q,\lambda k)}{(q-\lambda k)^2} - (\lambda \to \kappa\lambda) \right] \right), \qquad (18)$$

where H_1 is the particular combination of the scalars defined in eq. (2) playing the GPSDE game. Furthermore, a proper dimensional analysis of eq. (18) requires to cut the integration domain in its r.h.s. into two pieces by introducing some additional scale q_0^2 (of the order of Λ^2_{QCD}). Clearly, the external momentum is not the only relevant scale in the problem and Λ_{QCD}, without which it would not be understandable that the UV behaviour differs drastically from the IR one, must be taken into account. A careful dimensional analysis of the integrals extended over both domains, $q^2 > q_0^2$ and $q^2 < q_0^2$, is mandatory [1]. In the second one —and only there— we will initially use the common, convenient, but not really justified assumption of a power law behaviour of the propagators in the deep infrared:

$$F(k^2) \sim \left(\frac{k^2}{q_0^2}\right)^{\alpha_F}, \qquad G(k^2) \sim \left(\frac{k^2}{q_0^2}\right)^{\alpha_G}. \qquad (19)$$

We shall not repeat here the details of our scaling analysis of eq. (18)[3] and simply summarize our conclusions in the following 2 tables. It is often claimed, after the study of the GPDSE, that $2\alpha_F + \alpha_G = 0$. In fact, as can be seen in the next table 1, this results emerges only[4] after assuming $\alpha_F \neq 0$ and discarding (reasonably) $\alpha_F = -1$.

However, if $\alpha_F = 0$ other solutions are also compatible with GPDSE (see table 2).

Some recent lattice results seem to exclude the standard ($2\alpha_G + \alpha_F = 0$)-solution [1,2]. If one admits these results (lattice also discards $\alpha_F = -1$), then one is led to conclude that GPDSE implies $\alpha_F = 0$.

Furthermore, it was shown in ref. [7] that the r.h.s. of eq. (18) is the sum of two terms behaving, respectively, as $\lambda^{2\text{Min}(\alpha_F+\alpha_G+\alpha_\Gamma,1)}$ and λ^2 when $\lambda \to 0$. So, it behaves as λ^2 when $\alpha_F = 0$. Then, one can prove that for any κ there

Table 1. Constraints imposed by the GPDSE to the critical behaviour of ghost and gluon propagators for the case $\alpha_F \neq 0$. The second column shows the behaviour on λ ($\lambda \to 0$) of eq. (18)'s r.h.s., while the l.h.s. behaves as $(\lambda^2)^{-\alpha_F}$.

$\alpha_F \neq 0$		
$\alpha_F + \alpha_G$	r.h.s.	Constraint
> 1	λ^2	$\alpha_F = -1$
$= 1$	$\lambda^2 \log\lambda$	excluded
< 1	$(\lambda^2)^{\alpha_F+\alpha_G}$	$2\alpha_F + \alpha_G = 0$

Table 2. The same constraints analysed in table 1 but here for $\alpha_F = 0$. The l.h.s. of eq. (18) behaves now as the next-to-leading term of the deep infrared expansion of $F(q^2)$ (third column).

$\alpha_F = 0$		
$\alpha_F + \alpha_G$	r.h.s.	Constraint
> 1	λ^2	$F(q^2) = A + Bq^2$
$= 1$	$\lambda^2 \log\lambda$	$F(q^2) = A + Bq^2 \log q^2$
< 1	$(\lambda^2)^{\alpha_G}$	$F(q^2) = A + Bq^{2\alpha_G}$

[3] The analysis done in [1] missed some possible solutions (for instance, the case $\alpha_F = 0$, $\alpha_G < 1$) mainly because of the fact that we had rejected the possibility of non-analytic sub-dominant terms in the dressing functions.

[4] The regularity of the ghost-gluon vertex is also needed as was discussed in [1].

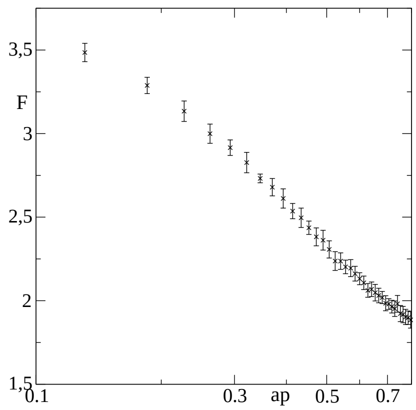


Fig. 1. $F(p)$ from a $SU(2)$ simulations on a 48^4 lattice at $\beta = 2.3$. (From ref. [7].)

is a value of λ and c such that

$$\left|\frac{1}{F(\lambda k)} - \frac{1}{F(\kappa^n \lambda k)}\right| \le c\frac{1-\kappa^{2n}}{1-\kappa^2}\lambda^2. \qquad (20)$$

So $F \to \infty$ when $\lambda \to 0$ is excluded because taking the limit of the above expression when $n \to \infty$ we should have $|\frac{1}{F(\lambda k)}| \le c\frac{1}{1-\kappa^2}\lambda^2$ and F would diverge as or more rapidly than $\frac{1}{\lambda^2}$ implying $\alpha_F \le -1$ in contradiction with the hypothesis $\alpha_F = 0$. Let us remark that $F \to 0$ is also excluded: equation (20) implies $|\frac{1}{F(\kappa^n \lambda k)}| \le |\frac{1}{F(\lambda k)}| + c\frac{1-\kappa^{2n}}{1-\kappa^2}\lambda^2$ and $\frac{1}{F(\kappa^n \lambda k)}$ cannot tend to infinity when $n \to \infty$. It should be emphasized that the dimensional analysis driving to eq. (20) is also valid if $F(q^2)$ is admitted to behave in a way other than a power. Thus, *if a leading power behaviour is discarded for the ghost dressing function, it has to be finite and* $\neq 0$ *in the IR limit.*

5 Conclusion

We derive from the Ward-Slavnov-Taylor identity for the three-gluon and ghost-gluon vertices, after assuming their regularity that gluon propagator diverges and ghost dressing function remains finite as the momentum goes to zero. A dimensional analysis of the GPDSE, provided that we trust the lattice results excluding $2\alpha_F + \alpha_G = 0$ [1,2] and $\alpha_F = -1$, leads to conclude independently that the ghost dressing function remains finite at zero momentum [7] (see tables 1, 2). Both GPDSE and WSTI constraints will offer compatible solutions provided that one admits non-analytic sub-leading terms for the low momentum expansion of dressing functions.

On the other hand, such a solution respecting WSTI and GPDSE constraints still match in the present picture of lattice knowledge about the IR behavior of propagators and vertices. The current simulations of ghost-gluon vertex seem to discard $2\alpha_F + \alpha_G = 0$ but those of ghost and gluon propagators cannot yet exclude or confirm the smooth divergences we propose as a way out [2,8,9] (as an example, see figs. 1 and 2, from refs. [7] and [1], respectively). A non-power behaviour (logarithmic, for instance) could be specially elusive for lattice extrapolations at infinite volume. Of course, new simulation results on bigger lattice volumes (or with twisted boundary conditions [10]) and careful extrapolations will be very welcome to dig into this matter.

This is a very interesting task to be acomplished, because either such a logarithmic (or similar) behaviour is found or one is led to conclude that the tensorial decomposition of ghost-gluon or three-gluon vertex admits non-regularities.

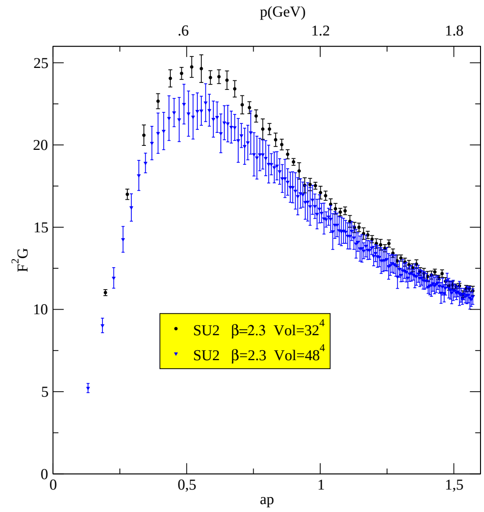


Fig. 2. F^2G from lattice simulation for $SU(2)$ (32^4 and $48^4, \beta_{SU(2)} = 2.3$) gauge groups. $2\alpha_F + \alpha_G = 0$ implies a constant in the infrared domain. (From ref. [1].)

References

1. Ph. Boucaud *et al.*, arXiv:hep-ph/0507104.
2. A. Sternbeck, E.M. Ilgenfritz, M. Muller-Preussker, A. Schiller, Nucl. Phys. Proc. Suppl. **153**, 185 (2006).
3. K.G. Chetyrkin, A. Retey, arXiv:hep-ph/0007088; A.I. Davydychev, P. Osland, O.V. Tarasov, Phys. Rev. D **54**, 4087 (1996); **59**, 109901 (1999)(E).
4. J.S. Ball, T.W. Chiu, Phys. Rev. D **22**, 2550 (1980).
5. Ph. Boucaud *et al.*, in preparation.
6. J.C. Taylor, Nucl. Phys. B **33**, 436 (1971); A.A. Slavnov, Theor. Math. Phys. **10**, 99 (1972).
7. Ph. Boucaud *et al.*, JHEP **0606**, 001 (2006).
8. F.D.R. Bonnet, P.O. Bowman, D.B. Leinweber, A.G. Williams, J.M. Zanotti, Phys. Rev. D **64**, 034501 (2001).
9. D. Becirevic, P. Boucaud, J.P. Leroy, J. Micheli, O. Pene, J. Rodriguez-Quintero, C. Roiesnel, Phys. Rev. D **61**, 114508 (2000).
10. T. Tok, K. Langfeld, H. Reinhardt, L. von Smekal, *The gluon propagator in lattice Landau gauge with twisted boundary*, PoS (LAT2005) 334 (2006) [arXiv:hep-lat/0509134].